

Towards Gravity solutions of AdS/CMT

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Abstract

In this short note, we have generalized and constructed gravity solutions with two “exponents” *a la* Kachru, Liu and Mulligan. The coordinate system that is used to construct the gravity solution is useful when b vanishes. It means we can describe the theory having only the temporal scale invariance apart from the combination of both temporal and spatial scale invariance. The two point correlation function of the scalar field in the mass less limit is computed in a special case that is $a/b = 2$.

Recently, it has been becoming very interesting to understand the gravity duals to 2+1 dimensional CFT's so as to understand the strongly coupled behavior of these CFT's, which may resemble some of the systems that we know in condensed matter theories (CMT). In this context several gravity solutions have been generated [1], [5]-[12] and more solutions need to be constructed so as to understand better the dual field theories at strong coupling.

The way to understand these systems is by constructing new gravity solutions with specific symmetry group. The scaling symmetry that we shall consider is

$$t \rightarrow \lambda^a t, \quad (x, y) \rightarrow \lambda^b (x, y), \quad r \rightarrow \frac{r}{\lambda}, \quad (1)$$

with the condition that we do not want to break the isotropicity along x and y directions. The constraint on the parameters a and b are $a \geq b$, and $ab \geq 0$. When $ab = 0$, we shall only take $b = 0$, but not a . The metric that remains invariant under this symmetry is

$$ds^2 = L^2 \left(-r^{2a} dt^2 + r^{2b} (dx^2 + dy^2) + \frac{dr^2}{r^2} \right). \quad (2)$$

For $a = b \neq 0$, the system has $SO(2, 3)$ symmetry group and is described as AdS_4 space time. This particular choice of the metric makes the time reversal manifest without the need to depend on the parity of the other directions. Moreover, it is useful to deal with for vanishing b . Which means we can study temporal scale invariance in this choice of coordinate system as opposed to the coordinate system used in [1].

For $a = 2$ and $b = 1$, the field theory action, which preserves the scaling symmetry, may be written as [2] and [3]

$$S_C = \frac{1}{2} \int d^2 x dt [(\partial_t \chi)^2 - K(\nabla^2 \chi)^2] \quad (3)$$

where K is a constant and describes a line of fixed points. If we break the rotational symmetry in the (x, y) plane of eq(2) then the dual field theory action could be [4]

$$S = S_C + S_{int} \quad (4)$$

where

$$\begin{aligned} S_C &= \frac{1}{2} \int d^2 x dt \left((\partial_t \chi)^2 - K[(\nabla^2 \chi)^2 + 4\sigma(\partial_x^2 \chi)(\partial_y^2 \chi)] \right), \\ S_{int} &= \int d^2 x dt \left(\frac{u}{4} [(\partial_x \chi)^4 + (\partial_y \chi)^4] + \frac{v}{2} (\partial_x \chi)^2 (\partial_y \chi)^2 \right) \end{aligned} \quad (5)$$

and the isotropic point corresponds to $\sigma = 0$ and $u = v$.

Now if we want to have a symmetry like eq(1) for generic a and b then the simplest dual field theory action, may be described as

$$S = \frac{1}{2} \int d^2 x dt [(\partial_t \chi)^\alpha - \tilde{K}(\nabla^2 \chi)^\beta] \quad (6)$$

where

$$\alpha = \frac{2b + a}{a} = 1 + \frac{2b}{a}, \quad \beta = \frac{2b + a}{2b} = 1 + \frac{a}{2b} \quad (7)$$

The action eq(6) has got the same structure as in eq(3) i.e. quadratic in fields and the computation of the 2-pt correlation function becomes Gaussian for a very specific choice of a and b that is $\frac{a}{b} = 2$.

Till now, the field theory action consistent with the scaling symmetry eq(1) that we have been discussing contains first derivative in time and second derivative in space coordinates. Now, the question arises: can we have an action that contains both second order derivatives in time and space coordinates? The answer to this is: yes¹.

$$S = \frac{1}{2} \int d^2x dt [(\partial_t^2 \chi)^\alpha - \hat{K}(\nabla^2 \chi)^\beta] \quad (8)$$

where

$$\alpha = \frac{2b+a}{2a} = \frac{1}{2} + \frac{b}{a}, \quad \beta = \frac{2b+a}{2b} = 1 + \frac{a}{2b} \quad (9)$$

and again for $a/b = 2$, the action is not any more quadratic in fields or for any other value of a/b . Hence, the most general action² consistent with the scaling symmetry and quadratic in fields is eq(6).

The metric eq(2) is non-singular and is well defined everywhere except at the origin $r = 0$, as it is not geodesically complete, for $b \neq 0$ in [1] and possibly be the same even for $b = 0$. However, the coordinate invariant quantities that are displayed in eq(19) says that we can make these quantities as small as we want by tuning the size i.e. L . The action that generates such a solution can be obtained from the action written in [1], which is a system containing gravity and fluxes of 2-form and 3-form type, as the relevant degrees of freedom

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} (R - 2\Lambda) - \frac{1}{4\kappa^2} \int (F_2 \wedge \star F_2 + F_3 \wedge \star F_3) + \frac{c}{2\kappa^2} \int B_2 \wedge F_2, \quad (10)$$

where $F_3 = dB_2$ and c is the topological coupling. The equations of motion that follows from it are

$$\begin{aligned} d \star F_2 &= -c F_3, \quad d \star F_3 = c F_2, \\ G_{MN} + \Lambda g_{MN} &= \frac{1}{2} \sum_{p=2,3} \frac{1}{p!} \left(p F_{M M_2 M_3 \dots M_p} F^{N M_2 M_3 \dots M_p} - \frac{1}{2} g_{MN} F_{M_1 M_2 \dots M_p} F^{M_1 M_2 \dots M_p} \right) \end{aligned} \quad (11)$$

where G_{MN} is the Einstein tensor. Ansatz to the fluxes consistent with the scaling symmetry are [1]

$$F_2 = \frac{AL^2}{r^{1-a}} dr \wedge dt, \quad F_3 = \frac{BL^3}{r^{1-2b}} dr \wedge dx \wedge dy \quad (12)$$

where for the 2-form it's only electric field and for 3-form it's magnetic type. It is easy to check that these fluxes obey Bianchi identities: $dF_2 = 0$, $dF_3 = 0$ even for $A=A(r)$ and $B=B(r)$.

¹ $S = \frac{1}{2} \int d^2x dt [(t \partial_t^2 \chi)^\alpha - \tilde{K}(\nabla^2 \chi)^\beta]$, but we do not know whether this kind of action is useful or not.

²As far as time reversal is concerned, the Lagrangian density is invariant but not the action, which changes sign by an overall factor and under space parity the action remains invariant.

Solving the equations of motion associated to these fluxes gives the restriction

$$\begin{aligned} cB(r) &= \frac{r^{1-2b}}{L} \frac{d}{dr}(A(r)r^{2b}), \\ cA(r) &= \frac{r^{1-a}}{L} \frac{d}{dr}(B(r)r^a) \end{aligned} \quad (13)$$

From the metric components as written in eq(2), we get

$$\begin{aligned} G_{tt} + \Lambda g_{tt} &= -r^{2a}(3b^2 + L^2\Lambda) \\ G_{xx} + \Lambda g_{xx} &= G_{yy} + \Lambda g_{yy} = r^{2b}(a^2 + b^2 + ab + L^2\Lambda) \\ G_{rr} + \Lambda g_{rr} &= \frac{2ab + b^2 + L^2\Lambda}{r^2} \end{aligned} \quad (14)$$

Finally, the equation of motion of the metric components gives

$$\begin{aligned} L^2(A^2 + B^2) &= -4(3b^2 + L^2\Lambda) \\ L^2(A^2 + B^2) &= 4(a^2 + b^2 + ab + L^2\Lambda) \\ L^2(B^2 - A^2) &= 4(2ab + b^2 + L^2\Lambda), \end{aligned} \quad (15)$$

where these equations arise from tt, xx and rr components, respectively. From these equations, we see that the right hand side of it are constants, which means the functions A(r) and B(r) better be constants. It means, eq(13) gives

$$\frac{A}{B} = \frac{a}{cL} = \frac{cL}{2b} \quad (16)$$

Now solving the equations eq(15) gives

$$\begin{aligned} A^2 &= \frac{2a(a-b)}{L^2}, \\ B^2 &= \frac{4b(a-b)}{L^2}, \\ \Lambda &= -\frac{a^2 + ab + 4b^2}{2L^2} \end{aligned} \quad (17)$$

It just follows that the coupling c is not any more arbitrary but is related to the exponents a and b as

$$c^2 L^2 = 2ab. \quad (18)$$

Some of the interesting properties about the solution eq(2)

$$\begin{aligned} R(\text{scalar curvature}) &= -\frac{2}{L^2}(a^2 + 2ab + 3b^2) = -\kappa^2 T_M^M, \\ R_{MN}R^{MN} &= \frac{2}{L^4}(a^4 + 2a^3b + 5a^2b^2 + 4ab^3 + 6b^4), \\ R_{MKNL}R^{MKNL} &= \frac{4}{L^4}(a^4 + 2a^2b^2 + 3b^4) \end{aligned} \quad (19)$$

are that the coordinate invariant quantities do depends on both the exponents a and b and T_M^M is the trace of the energy-momentum tensor.

Let us do some change of coordinates³ and try to make contact with [1]

$$r^b = \rho, \quad (t, x, y) \rightarrow \frac{1}{b}(t, x, y), \quad L \rightarrow bL. \quad (20)$$

This change of coordinates makes sense only when $b \neq 0$ and under this change we get the metric eq(2) as

$$ds^2 = L^2[-\rho^{2\frac{a}{b}}dt^2 + \rho^2(dx^2 + dy^2) + \frac{d\rho^2}{\rho^2}]. \quad (21)$$

This form of the metric coincides with the one written in [1] by defining $z = a/b$.

We can do a similar change of coordinates

$$r^a = \tilde{\rho}, \quad (t, x, y) \rightarrow \frac{1}{a}(t, x, y), \quad L \rightarrow aL, \quad (22)$$

and this makes sense when $a \neq 0$. Under this change we get the metric eq(2) as

$$ds^2 = L^2[-\tilde{\rho}^2dt^2 + \tilde{\rho}^{2\frac{b}{a}}(dx^2 + dy^2) + \frac{d\tilde{\rho}^2}{\tilde{\rho}^2}]. \quad (23)$$

Let us recall the solutions that we have in eq(17), from this it follows that in order to have real fluxes we can only take b to vanish but not a . The other conditions are $ab \geq 0$, and $a \geq b$.

The metric written in eq(2), in the (r, t, x, y) coordinate system is a good one as one can consider the $b = 0$ case but not in the (ρ, t, x, y) coordinate system that is used to write eq(21). However, if we want to concentrate on the situations when b do not vanishes, then any of coordinate system is good.

The physics of eq(2) is that we can study a combination of both spatial and temporal scale invariance as well as temporal scale invariance independently, whereas if we use the coordinate system that is written in eq(21), then study of only temporal scale invariance is not possible. It is important to note that in none of coordinate system that we know of to describe only the spatial scale invariance. It probably makes sense to say that in order to describe only temporal scale invariance, we need to have two exponents in the metric rather than one and we cannot study only the spatial scale invariance because of the reality constraint on the fluxes.

In the next section we shall study some of the properties of the operators in the dual field theory using the generalized form of the gauge/gravity correspondence in which we shall keep b to be arbitrary, but while studying the 2pt correlation function of operators that are dual to scalars, we shall use a specific choice to a and b .

³Thanks to Alex Maloney for a useful correspondence.

1 Field theory observable

Let us consider a real scalar field ϕ of mass m that is propagating in the background of eq(2), which in the $u = 1/r$ coordinate is

$$ds^2 = L^2 \left(-\frac{dt^2}{u^{2a}} + \frac{dx^2 + dy^2}{u^{2b}} + \frac{du^2}{u^2} \right) \quad (24)$$

The equation of motion for a minimally coupled scalar field ϕ is

$$\partial_u^2 \phi - \frac{a+2b-1}{u} \partial_u \phi - [w^2 u^{2(a-1)} + (k_x^2 + k_y^2) u^{2(b-1)} + \frac{(mL)^2}{u^2}] \phi = 0 \quad (25)$$

In order to understand the generalized form of the AdS/CFT dictionary in this case, we need to find the relation between the operator dimension Δ and the mass of the field, m , where, the field ϕ is dual to an operator of dimension Δ for which

$$\begin{aligned} \Delta(\Delta - a - 2b) &= m^2 L^2, \\ \Delta_{\pm} &= b + \frac{a}{2} \pm \sqrt{(b + \frac{a}{2})^2 + m^2 L^2}. \end{aligned} \quad (26)$$

Δ_+ and Δ_- are the two roots of the first equation with $\Delta_+ \geq \Delta_-$.

On requiring the finiteness of the Euclidean action of the scalar field, as is done in [13] for asymptotically AdS space time, imposes the restriction that if the mass of the scalar field stays

$$(mL)^2 > 1 - \left(\frac{a+2b}{2} \right)^2 \quad (27)$$

above this bound then only Δ_+ branch is allowed, whereas if the scalar field has a mass that stays between

$$-\left(\frac{a+2b}{2} \right)^2 < (mL)^2 < 1 - \left(\frac{a+2b}{2} \right)^2 \quad (28)$$

then both Δ_- and Δ_+ branches are allowed.

The analogue of Breitenlohner-Freedman bound [14] for this case is

$$(mL)^2 < -\left(\frac{a+2b}{2} \right)^2 \quad (29)$$

and if the mass stays below this bound then there is an instability in the system.

It is well known that in a CFT the two point correlation of an operator with dimension Δ goes as

$$\langle \mathcal{O}(x) \mathcal{O}(0) \rangle \sim \frac{1}{|x|^{2\Delta}} \quad (30)$$

However, for a non-relativistic CFT the two point correlator instead of just going like a power law falloff it can get dressed by an exponential falloff [6]. Whatever be the case, it's for sure that there will be a power law falloff, which follows from the translational and rotational symmetry in the spatial directions. Upon assuming that is the case it just follows trivially that the two point correlation function depends on the parameters on which Δ depends.

In our case Δ depends on the parameter a and b in the combination $a + 2b$, which means there will be two exponents in the 2-pt correlation function. For $b \neq 0$, we can rewrite the expression to Δ_{\pm} for which Δ_{\pm}/b depends on a and b in a specific way that is a/b with a redefinition to L . But, unfortunately, there do not look like the presence of any phase transitions.

According to the AdS/CFT prescription [15], [16] the correlation function of operators is evaluated by differentiating the on shell value of the action with respect to a specific boundary values of the bulk field.

For the minimally coupled scalar field with generic values of a and b , it is not easy to compute the correlation function, analytically. However for a specific choice of $a/b = 2$, one can solve eq(25). If we recall from eq(6), the value of a/b for which the action is quadratic in fields is $a/b = 2$.

For this particular choice of a/b , the normalized solution is

$$G(u, \vec{\mathbf{k}}, \omega) = c_1 \times 2^{\frac{1}{2}[1 + \frac{\sqrt{4b^2 + m^2 L^2}}{b}]} \times e^{-\frac{\omega}{2b} u^{2b}} \times u^{2b + \frac{\sqrt{4b^2 + m^2 L^2}}{b}} \times U\left(\frac{\vec{\mathbf{k}}^2 + 2b\omega + 2\omega\sqrt{4b^2 + m^2 L^2}}{4b\omega}, 1 + \frac{\sqrt{4b^2 + m^2 L^2}}{b}, \frac{\omega}{b} u^{2b}\right), \quad (31)$$

where c_1 is a normalization constant and to be fixed by the condition

$$G(u \rightarrow \epsilon, \vec{\mathbf{k}}, \omega) = 1 \quad (32)$$

and $U(a, b, z)$ is the confluent hypergeometric function of the second kind.

Let us recall that the action of a scalar field

$$S = -\frac{1}{2} \int d^3x du \sqrt{g} (g^{MN} \partial_M \phi \partial_N \phi + m^2 \phi^2) \quad (33)$$

which upon using equation of motion can be re-written as

$$S = -\frac{1}{2} \int d^3x [\sqrt{g} (g^{uu} \phi \partial_u \phi + m^2 \phi^2)]_{\epsilon}^{\infty} \quad (34)$$

where we have introduced a regulator ϵ to regulate the ultraviolet divergences. The last equation in momentum space can be re-written, by introducing the sources at the boundary that is $\phi(u, k) = G(u, k) \phi(0, k)$, where we use a condensed notation k_{μ} to represent the three vector, $k = (u, \vec{\mathbf{k}})$.

$$S = \int d^2k dw \phi(0, -k) \mathcal{F}(k) \phi(0, k) \quad (35)$$

and the flux factor is

$$-\frac{1}{2} \mathcal{F}(k) = [G(u, -k) \sqrt{g} g^{uu} \partial_u G(u, k)]_{\epsilon}^{\infty} \quad (36)$$

Now the two point correlation function of operator \mathcal{O} associated to the dual of a scalar field is calculated by differentiating twice the action of the scalar field with respect to the source $\phi(0, k)$

$$\langle \mathcal{O}(-k) \mathcal{O}(k) \rangle = \mathcal{F}(k) \quad (37)$$

In order to proceed further let us restrict ourselves to the mass less sector and in this case the functions that appear in the two point correlator are

$$\begin{aligned}
G(u, k) &= 1 - \frac{\omega}{2b} u^{2b} + \left(\alpha \frac{\omega^2}{2b^2} + \frac{\omega^2}{8b^2} - \right. \\
&\quad \left. \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} \frac{\omega^2}{4b^2} [-3 + 4\gamma + 2\psi(2+\alpha) + 2\log(\frac{w}{b}) + 4b \log u] \right) + \mathcal{O}(u)^{6b} \\
\partial_u G(u, k) &= u^{4b-1} \left[-\omega u^{-2b} + \frac{2\alpha\omega^2}{b} + \frac{\omega^2}{2b} - \right. \\
&\quad \left. \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} \frac{\omega^2}{b} [-4 + 4\gamma + 2\psi(2+\alpha) + 2\log(\frac{\omega}{b}) + 4b \log u] + \mathcal{O}(u)^{6b-1} \right] \\
\sqrt{g} g^{uu} &= L^2 u^{1-4b}
\end{aligned} \tag{38}$$

where

$$\alpha = -\frac{1}{2} + \frac{\vec{\mathbf{k}}^2}{4b\omega}, \tag{39}$$

γ is the Euler-Mascheroni constant and $\psi(x)$ is the digamma function. using all these ingredients into eq(37) we find

$$< \mathcal{O}(-k) \mathcal{O}(k) > = \mathcal{F}(k) = -\frac{L^2}{2} \vec{\mathbf{k}}^2 \omega + \frac{L^2}{b^3} (\vec{\mathbf{k}}^4 - b^2 \omega^2) [-2 + \log(\omega) - \psi(\frac{3}{2} + \frac{\vec{\mathbf{k}}^2}{4b\omega})]. \tag{40}$$

Summarizing all this, we have presented a better coordinate system than [1] to handle the case for which b vanishes. Generically, the constraint on the parameters a and b are $ab \geq 0$ and $a \geq b$. The situation when $ab = 0$, we take b to vanish but not a , in order to have real fluxes. It means that the coordinate system that we used to write eq(2), is better than the coordinate system used to write eq(21), in the sense that we can describe the temporal scale invariance along with the combination of both temporal and spatial scale invariance, independently. However, if want to have an action in the dual field theory to be quadratic (or any other value) in fields means, $b \neq 0$ and it suggests that either of the coordinate system is good.

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